Structure and instability of high-density equations for traffic flow

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Similar to the treatment of dense gases, fluid-dynamic equations for the dynamics of congested vehicular traffic are derived from Enskog-like kinetic equations. These contain additional terms due to the anisotropic vehicle interactions. The calculations are carried out up to Navier-Stokes order. A linear instability analysis indicates an additional kind of instability compared to previous macroscopic traffic models. The relevance for describing granular flows is outlined. [S1063-651X(98)06104-2]

An efficient infrastructure is an essential precondition for every industrialized country. Therefore, the considerable deterioration of the traffic situation on ''freeways'' during the past decade is a serious problem. Not only does impeded traffic cause economic losses of many billions of dollars each year, it also produces serious ecological damages. Thus great efforts have been made to develop methods for traffic optimization and forecasts, for which reliable traffic simulations are necessary. Consequently, research on traffic dynamics recently has become a very important topic. In recent years, numerous results have been published on microscopic models [1–4], including cellular automata models [2] and molecular-dynamics-like models [3,4], kinetic models [4–10], and macroscopic (fluid-dynamic-like) models [11,4–10], aiming at an understanding of stop and go traffic. The topic is related to the fields of nonlinear dynamics [11], phase transitions [1–4,11], and stochastic processes [1,2].

Macroscopic traffic models are suitable not only for on-line simulations of traffic networks, but also for analytical investigations. Some very recent publications proposed the derivation of *realistic* traffic equations from the ''microscopic'' dynamics of driver-vehicle units via a kinetic approach [4–10]. However, for the following reasons, these attempts have been only partly successful. Either the models treat the vehicles like pointlike objects [5–7], in which case the resulting macroscopic models are only valid for free traffic flow, or the models take into account the finite-space requirements of vehicles [4,8–10]. However, the calculations were carried out up to *Euler* order only, based on the (zeroth-order) approximation of local equilibrium. This assumes that the *form* of the equilibrium velocity distribution remains unchanged in dynamic situations, but it is given by the *local* values of the density $\rho(r,t)$, average velocity $V(r,t)$, etc. It will be demonstrated that corresponding traffic models are not even valid in linear approximation since their linear stability analysis gives totally misleading results. To solve this problem, we must calculate the (first-order) Navier-Stokes corrections of the macroscopic traffic equations, which take into account the structural change of the velocity distribution in inhomogeneous traffic situations.

For ordinary gases, the Navier-Stokes terms (transport terms) are calculated from the kinetic equation by means of the Chapman-Enskog method [12]. An approximate but more intuitive method is based on the relaxation time approximation [13,14] (see below). For kinetic *traffic* models

the situation is more involved since the interaction term does not vanish in equilibrium situations. Now this problem has been solved. In the following, we will show how to derive realistic traffic equations that include corrections due to vehicular space requirements *and* Navier-Stokes terms. Since these nontrivial corrections change the *structure* of the equations, they cause an additional instability. For the same reason, the low-density regime does not allow an extrapolation to situations at high densities. The method presented is also relevant for understanding instabilities in granular flows since granular collisions are also not energy conserving. Recently, many publications tried to tackle these interesting problems with fluid-dynamic equations derived from the associated kinetic equation [15–17]. However, most of them are restricted to Euler order or the low-density regime [16,17], thereby neglecting relevant sources of stability and instability. Probably for this reason, these approaches have not been fully successful in describing the formation of density waves in sand that is falling through a vertical pipe [17,18].

Now we discuss the model. Since the number of vehicles on a (for simplicity, circular) freeway is conserved, the kinetic traffic equation for the phase space density $\widetilde{\rho}(r,v,t)$ of vehicles with velocity $v=d_t r$ at place r and time t has the form of a continuity equation with a sink or source term

$$\partial_t \widetilde{\rho} + \partial_r(\widetilde{\rho}v) + \partial_v(\widetilde{\rho}\ d_t v) = (\partial_t \widetilde{\rho})_{SS}. \tag{1}$$

As usual, we will assume the acceleration law $d_t v = (V_0 - v)/\tau$, where τ denotes a density-dependent acceleration time and V_0 the desired velocity, which is assumed to be the same for all vehicles here (the case of a speed limit). The sink or source term $(\partial_t \widetilde{\rho})_{SS}$ originates from sudden (non-differentiable) velocity changes. It splits up into a velocity-diffusion term due to fluctuations of the acceleration behavior (''imperfect driving'') and an interaction term

$$(\partial_t \widetilde{\rho})_{SS} = \partial_v^2(\widetilde{\rho}D) + (\partial_t \widetilde{\rho})_{int}. \tag{2}$$

The interaction term reflects sudden deceleration processes. In analogy to the Enskog theory of dense gases [19,20] and granular media [15], but with an interaction law typical for vehicles [8,4], it is of the form

$$(\partial_t \widetilde{\rho})_{int} = (1-p)\chi(r+l,t)\mathcal{B}(v), \tag{3}$$

with the Boltzmann-like interaction function

$$\mathcal{B}(v) = \int_{w>v} dw(w-v)\widetilde{\rho}(r,w,t)\widetilde{\rho}(r+s,v,t) \quad (4a)$$

$$-\int_{v>w} dw(v-w)\widetilde{\rho}(r,v,t)\widetilde{\rho}(r+s,w,t). \quad (4b)$$

According to this, the phase-space density $\widetilde{\rho}(r,v,t)$ increases due to deceleration of vehicles with velocities $w > v$, which cannot overtake vehicles with velocity v. The density-dependent probability of immediate overtaking is represented by p. A decrease of the phase-space density $\widetilde{\rho}(r,v,t)$ is caused by interactions of vehicles with velocity v with slower vehicles driving with velocities $w < v$. 9 The corresponding interaction rates are proportional to the relative velocity $|v-w|$ and to the phase-space densities of both interacting vehicles. By $s(V) = l_0 + l(V) \times [\approx (\text{vehicle length}) + (\text{safe distance})]$ it is taken into account that the distance of interacting vehicles is given by their velocity-dependent space requirements. These cause an increase of the interaction rate, which is described by the pair-correlation function $\chi(r) = [1 - \rho(r,t)s]^{-1}$ at the "interaction point" $r+l$. A more detailed discussion of the above kinetic traffic model is presented elsewhere [4,8]. Describing the individual acceleration behavior via $d_t v = (V_0 - v)/\tau$ and introducing a velocity-diffusion term as the source of velocity variations improves the original approach by Prigogine and Herman [4,5], which assumes a relaxation of the actual phase-space density to a desired one [6].

Now we will focus on the macroscopic equations for the spatial density $\rho(r,t) = \int dv\, \widetilde{\rho}(r,v,t)$, the average velocity $V(r,t) = \int dv\, v\,\widetilde{\rho}(r,v,t)/\rho(r,t)$, and the velocity variance $\Theta(r,t) = \int dv [v - V(r,t)]^2 \widetilde{\rho}(r,v,t)/\rho(r,t)$. These are obtained by multiplying the kinetic equation by v^k, integrating with respect to v, and a number of straightforward calulations [4,8]. In order to underline the crucial results of this paper, we will first discuss the case of *negligible* space requirements $[s,l,l_0 \ll 1/\rho(r,t)]$, in which the macroscopic traffic equations read

$$\partial_t \rho + V\partial_r \rho = -\rho\partial_r V, \quad (5)$$

$$\partial_t V + V\partial_r V = -1/\rho\,\partial_r \mathcal{P} + (V_0 - V)/\tau - (1-p)\mathcal{P}, \quad (6)$$

$$\partial_t \Theta + V\partial_r \Theta = -2\mathcal{P}/\rho\,\partial_r V - 1/\rho\,\partial_r \mathcal{J} + 2(D-\Theta/\tau)$$
$$-(1-p)\mathcal{J}. \quad (7)$$

Here $\mathcal{P} = \rho\Theta$ denotes the "pressure" and $\mathcal{J}(r,t) = \rho(r,t)\Gamma(r,t) = \int dv [v-V(r,t)]^3 \widetilde{\rho}(r,v,t)$ the flow of velocity variance. Equation (5) is the expected continuity equation for the density. In comparison to the conventional Euler equations for ordinary gases, the velocity equation (6) and the variance equation (7) contain two additional terms, each of which breaks momentum and energy conservation. The respective last terms result from the anisotropic vehicle interactions, while the previous terms reflect acceleration behavior and velocity fluctuations.
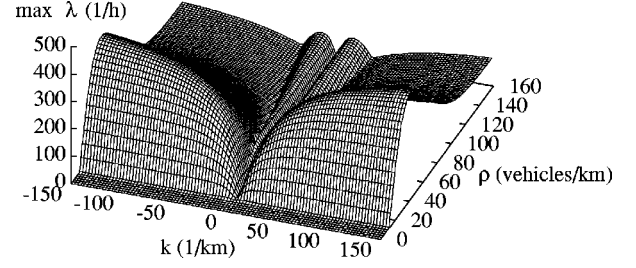

FIG. 1. The instability diagram of the Navier-Stokes-like traffic model for pointlike vehicles ($s = l = l_0 = 0$) indicates that traffic flow would be unstable above a certain critical density, surprisingly even at large wave numbers k.

It can be shown that the kinetic traffic equation has the Gaussian equilibrium solution $\widetilde{\rho}_0(v) = \rho(2\pi\Theta)^{-1/2}\exp[-(v-V)^2/2\Theta]$, which additionally fulfills the implicit equilibrium relations $V = V_0 - \tau(1-p)\rho\chi\Theta$ and $\Theta = D\tau$. If the local values $\rho(r,t)$, $V(r,t)$, and $\Theta(r,t)$ are inserted instead, we obtain the Euler approximation. This leads to $\mathcal{J} = \rho\Gamma \approx 0$ [4,21]. However, in *inhomogeneous* traffic situations the form of the velocity distribution $P(v;r,t) = \widetilde{\rho}(r,v,t)/\rho(r,t)$ changes due to the finite adaptation time τ_0 that is needed to reach local equilibrium. In the relaxation-time approximation [5,13,14] we find the Navier-Stokes correction $\Gamma = -3\sqrt{\pi\Theta}/[(1-p)\rho]\partial_r\Theta$. The corresponding instability diagram is depicted in Fig. 1 for the following model functions approximating empirical results [4,8]: $\tau(\rho) = (8\,\text{s})/\{0.97\exp[-\rho/(16\ \text{km}^{-1})] + 0.03\}$, $p(\rho) = \exp[-\rho/(16\ \text{km}^{-1})]$, and $D = 0.03V^2/\tau(\rho)$.

The instability diagram is obtained by (i) assuming a small periodic perturbation $\delta g(r,t) = g_0\exp[ikr + (\lambda + i\omega)t]$ of the macroscopic traffic quantities $g \in \{\rho, V, \Theta\}$ relative to the stationary and spatially homogeneous equilibrium solution $g_e(\rho)$ (g_0 being the amplitude, k the wave number, λ the growth rate, and ω the frequency of the perturbation), (ii) inserting $g(r,t) = g_e + \delta g(r,t)$ into the macroscopic traffic equations, (iii) neglecting quadratic terms in the small perturbations $\delta g(r,t) \ll g_e$, and (iv) determining the three complex eigenvalues $\widetilde{\lambda} = \lambda + i\omega$ of the linearized equations dependent on ρ and k. Equilibrium traffic flow is unstable, giving rise to the well-known phenomenon of *stop and go traffic* [4,11], if at least one of the growth rates is positive, i.e., max $\lambda > 0$. Therefore, the instability diagram shows max$\lambda(k,\rho)$ if this is greater than zero and 0 otherwise.

It is interesting to compare Fig. 1 with the instability diagram in Fig. 2, which corresponds to the Euler approximation with $\Gamma = 0$. This shows that the curious maxima in the middle of Fig. 1 are an effect of the deformation of the Gaussian velocity distribution in inhomogeneous situations. Since they originate from the terms containing Γ, they are directly connected with the dynamic variance equations. Therefore, they could not be discovered in previous traffic models that eliminated $\Theta(r,t)$ by means of approximations of the kind $\Theta(r,t) \approx \Theta^e(\rho(r,t), V(r,t))$ [4,8].

Apparently, the Navier-Stokes corrections do not cause a stability of equilibrium traffic flow with respect to perturbations of large wave numbers k (i.e., small wavelengths $2\pi/k$). This surprising result is a serious problem for a numerical solution of the above equations. It comes from the
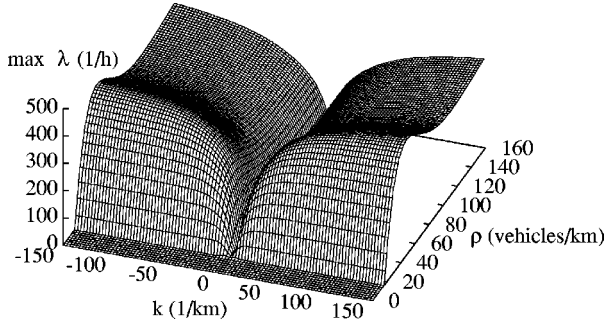
FIG. 2. Same as Fig. 1, but for macroscopic traffic equations calculated up to Euler order only, thereby neglecting the deformation of the velocity distribution in inhomogeneous traffic situations.

FIG. 3. Same as Fig. 2, but with consideration of vehicular space requirements ($l_0 = 1/\rho_{\max}$, $l = 0.8$ sV). In agreement with empirical findings [21], the results predict that equilibrium traffic flow is stable up to 12 vehicles per kilometer and lane, at extreme densities, and at high wave numbers (small wave lengths).

fact that (shear) viscosity terms are missing due to the spatial one-dimensionality of traffic flow. This problem vanishes when corrections due to vehicular space requirements are taken into account ($l_0 = 1/\rho_{\max}$, $l = 0.8$ s V). One would not expect this since, for ordinary gases, the *structure* of the fluid-dynamic equations does not change at high densities. The only changes are the constitutive relations for $\mathcal{P}$ and $\mathcal{J}$ [19,4]. However, in the case of traffic dynamics the situation is completely different. Since the anisotropic vehicle interactions do not fulfill momentum and energy conservation, they lead to contributions that cannot be absorbed by modified functions $\mathcal{P}$ and $\mathcal{J}$. Whereas the vehicle density still obeys the continuity equation (5), the structure of the velocity and variance equations changes considerably [$\gamma = (1-p)\chi$]:

$$\partial_t V + V \partial_r V = -[1/\rho + \gamma s(1+\rho\chi l)]\Theta \partial_r \rho$$
$$+ \gamma\rho(2s\sqrt{\Theta/\pi} - \rho\Theta\chi l^2/V)\partial_r V$$
$$-[1 + \gamma\rho s/2]\partial_r\Theta - \gamma s(s/2 + \rho\chi l^2/2)\Theta\partial_r^2\rho$$
$$+ \eta/\rho\partial_r^2 V - \gamma\rho s^2/4\partial_r^2\Theta + (V_0 - V)/\tau - \gamma\rho\Theta,$$
$$(8)$$

$$\partial_t\Theta + V\partial_r\Theta = -[2 + \gamma\rho s]\Theta\partial_r V + 2\gamma\rho s\sqrt{\Theta/\pi}\partial_r\Theta$$
$$- \gamma\rho s^2\Theta/2\partial_r^2 V + \gamma\rho s^2\sqrt{\Theta/\pi}\partial_r^2\Theta$$
$$+ 2(D - \Theta/\tau).$$
$$(9)$$

This result is valid up to Euler order. It has been obtained by evaluating the kinetic equation on the assumption of a Gaussian velocity distribution $P_0(v;r,t)$ and second-order Taylor expansion of the functions $\widetilde{\rho}$ and χ with respect to s and l around r, thereby neglecting products of partial derivatives. Figure 3 shows that the finite-space requirements of vehicles cause the desired stability of equilibrium traffic flow with respect to perturbations of small wavelengths. This comes from the finite viscosity coefficient $\eta = \gamma\rho^2[s^2\sqrt{\Theta/\pi} - \rho\Theta\chi l^3/2V]$.

Nevertheless, the result is not even correct in the linear approximation since inhomogeneous traffic again changes the form of the velocity distribution. To calculate the Navier-Stokes corrections, we must derive an equation for the small deviation $\delta\widetilde{\rho}(r,v,t)$ from $\widetilde{\rho}_0(r,v,t) = \rho(r,t)P_0(v;r,t)$, which is caused by inhomogeneities $\partial_r\rho$, $\partial_r V$, and $\partial_r\Theta$. This has been done by means of the relaxation time approxima-
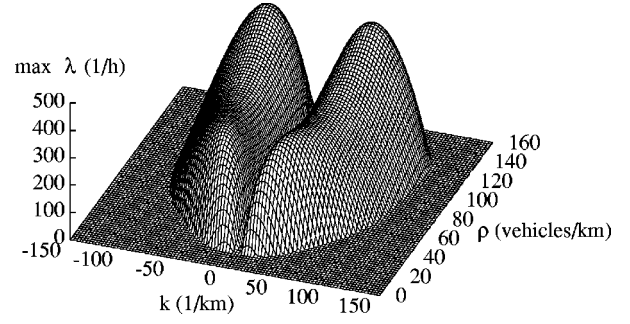
tion [5,13,14], which assumes that (i) the deformation of the local equilibrium distribution $P_0(v;r,t)$ is caused by the interaction term, (ii) the nonequilibrium corrections of the latter can be adiabatically approximated by $-\delta\widetilde{\rho}(r,v,t)/\tau_0$, where $-1/\tau_0$ denotes the slowest eigenvalue of the linearized interaction operator, and (iii) $1/\tau_0$ is of the order of the vehicular interaction rate

$$\frac{1}{\tau_0(r,t)} = (1-p)\chi(r+l,t)/\rho(r,t)\int dv$$
$$\times \int_{w<v} dw |v-w|\widetilde{\rho}(r,v,t)\widetilde{\rho}(r+s,w,t).$$
$$(10)$$

The finally resulting relation is, with $\delta v = v - V$,

$$\delta\widetilde{\rho}(r,v,t) \approx \widetilde{\rho}_0(r,v,t)\tau_0(3\delta v/\Theta - \delta v^3/\Theta^2)/2\partial_r\Theta$$
$$+ \tau_0(1-p)\chi(r+l,t)\mathcal{B}(v) - P_0(r,v,t)\tau_0\delta v/\Theta$$
$$\times (1-p)\chi(r+l,t)\int dv\,\delta v\,\mathcal{B}(v)$$
$$- \frac{P_0(r,v,t)\tau_0}{2\Theta^2}(\delta v^2 - \Theta)$$
$$\times (1-p)\chi(r+l,t)\int dv\,\delta v^2\mathcal{B}(v).$$
$$(11)$$

With $\widetilde{\rho}(r,v,t) \approx \rho_0(r,v,t) + \delta\widetilde{\rho}(r,v,t)$, it follows that the quantities $\rho(r,t)$, $V(r,t)$, and $\Theta(r,t)$, which are taken into account by the Gaussian approximation $P_0(v;r,t)$, are not corrected by $\delta\widetilde{\rho}(r,v,t)$. However, for the third central velocity moment Γ we obtain, instead of $\Gamma \approx 0$,

$$\Gamma = -3\sqrt{\pi\Theta}/[(1-p)\rho\chi]\partial_r\Theta + s\Theta\partial_r V + s^2\Theta/2\partial_r^2 V$$
$$- 3s\sqrt{\pi\Theta}/2\partial_r\Theta - 3s^2\sqrt{\pi\Theta}/4\partial_r^2\Theta,$$
$$(12)$$

which becomes different from zero in inhomogeneous traffic situations. This causes the additional contribution

$$+ \rho s/(6\sqrt{\pi\Theta})\partial_r\Gamma + \rho s^2/(12\sqrt{\pi\Theta})\partial_r^2\Gamma \qquad (13)$$
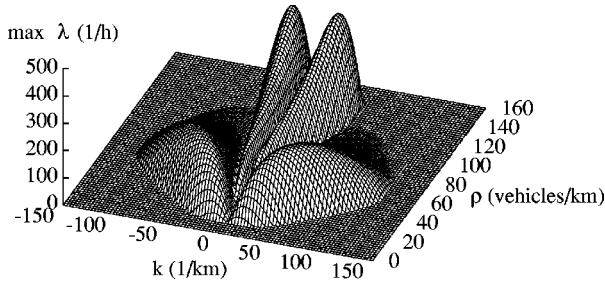
to the velocity equation (8) and the extra term

FIG. 4. Same as Fig. 3, but for macroscopic traffic equations including vehicular space requirement *and* Navier-Stokes corrections in order to obtain valid results. The instability diagram is now divided into two separate humps in each half plane, indicating two different kinds of instability.

$$-\Gamma[\rho + s\partial_r\rho + s^2/2\partial_r^2\rho - \rho s/\sqrt{\pi\Theta}\,\partial_r V - \rho s^2/(2\sqrt{\pi\Theta})\partial_r^2 V]$$

$$-\rho s/2\partial_r\Gamma - \rho s^2/4\partial_r^2\Gamma \tag{14}$$

to the variance equation (9). Together with the continuity equation (5), the resulting equations are the desired macroscopic traffic equations for high densities. The related instability diagram is depicted in Fig. 4 and indicates two different kinds of instabilities.

In summary, we have found several significant results, which are not sensitive to the particular choice of the parameters or to different variations of the model. (i) A consistent traffic model needs to take into account vehicular space requirements as well as Navier-Stokes terms in order to allow a realistic description of traffic instabilities. (ii) Treating vehicles in a pointlike manner, the stability of equilibrium traffic flow with respect to perturbations of small wavelengths is not correctly described, even if Navier-Stokes terms are included. (iii) This problem vanishes when corrections due to the finite-space requirements of vehicles are considered. That is, the macroscopic traffic equations for low densities do not allow an extrapolation to the traffic dynamics at high densities. (iv) The Navier-Stokes terms are responsible for a subdivision of the instability region into separate areas. These belong to different eigenvalues. (v) Whereas the instability diagrams of traffic models, which consist of a density and a velocity equation only, typically show two relevant humps [4,8], the two additional (narrow) humps of the above model are related to the dynamic variance equation. Therefore, the dynamic variance equation gives rise to a different kind of traffic instability. A more detailed discussion of the applied relaxation time approximation as well as of equations, simulations, and results for the nonlinear regime of traffic dynamics will be given in a forthcoming paper [14].

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